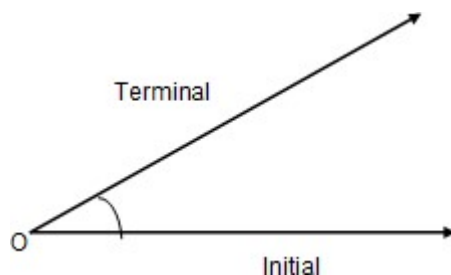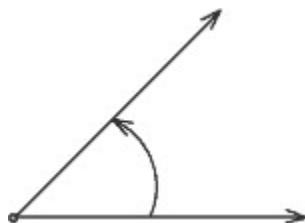# MATHEMATICS

Chapter 3: TRIGONOMETRIC FUNCTION

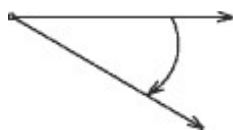# TRIGONOMETRIC FUNCTION

Top Concepts

1. An angle is a measure of rotation of a given ray about its initial point. The original position of the ray before rotation is called the initial side of the angle, and the final position of the ray after rotation is called the terminal side of the angle. The point of rotation is called the vertex.

2. If the direction of rotation is anticlockwise, then the angle is said to be positive, and if the direction of rotation is clockwise, then the angle is negative.
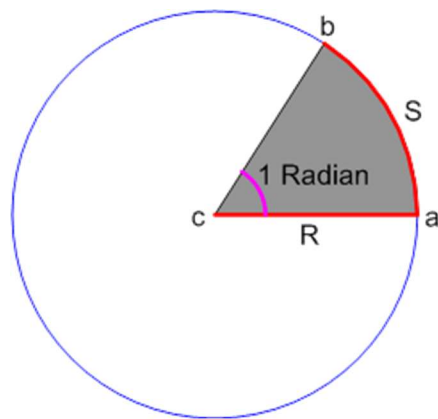

Positive angle—Anticlockwise

Negative angle—Clockwise

3. If a rotation from the initial side to terminal side is $\left(\frac{1}{360}\right)^{th}$ of a revolution, then the angle is said to have a measure of one degree. It is denoted by 1°.

4. A degree is divided into 60 minutes, and a minute is divided into 60 seconds. One sixtieth of a degree is called a minute and is written as 1', and one sixtieth of a minute is called a Second and is written as 1".
Thus, 1° = 60' and 1' = 60".

5. The angle subtended at the centre by an arc of length 1 unit in a unit circle is said to have a measure of 1 radian.

6. Basic trigonometric ratios
   Consider the following triangle:

   i. $Sine\theta = \dfrac{Perpendicular}{Hypotenuse}$

   ii. $Cosine\theta = \dfrac{Base}{Hypotenuse}$

   iii. $Tangent\theta = \dfrac{Perpendicular}{Base}$

   iv. $Cosecant\theta = \dfrac{Hypotenuse}{Perpendicular}$

   v. $Secant\theta = \dfrac{Hypotenuse}{Base}$

vi. $\text{Cotangent}\theta = \dfrac{\text{Base}}{\text{Perpendicular}}$

7. Some trigonometric identities

i. $\sin\theta = \dfrac{1}{\cos ec}$ or $\cos ec\theta = \dfrac{1}{\sin\theta}$

ii. $\cos\theta = \dfrac{1}{\sec\theta}$ or $\sec\theta = \dfrac{1}{\cos\theta}$

iii. $\tan\theta = \dfrac{1}{\cot}$ or $\cot\theta = \dfrac{1}{\tan\theta}$

iv. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

v. $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

vi. $\sin^2\theta + \cos^2\theta = 1$

vii. $1 + \tan^2\theta = \sec\ \theta$

viii. $1 + \cot^2\theta = \cos ec\ \theta$

8. If a point on the unit circle is on the terminal side of an angle in the standard position, then the sine of such an angle is simply the y-coordinate of the point and the cosine of the angle is the x-coordinate of that point.

9. All the angles which are integral multiples of $\frac{\pi}{2}$ are called quadrantal angles. Values of quadrantal angles are:

    cos 0 = 1, sin 0 = 0

    $\cos \frac{\pi}{2} = 0$, $\sin \frac{\pi}{2} = 1$

    cos π = -1, sin π = 0

    $\cos \frac{3\pi}{2} = 0$, $\sin \frac{3\pi}{2} = -1$

    cos2 π = 1, sin2 π = 0

10. Even function: A function f(x) is said to be an even function if f(-x) = f(x) for all x in its domain.
11. Odd function: A function f(x) is said to be an odd function if f(-x) = -f(x) for all x in its domain.
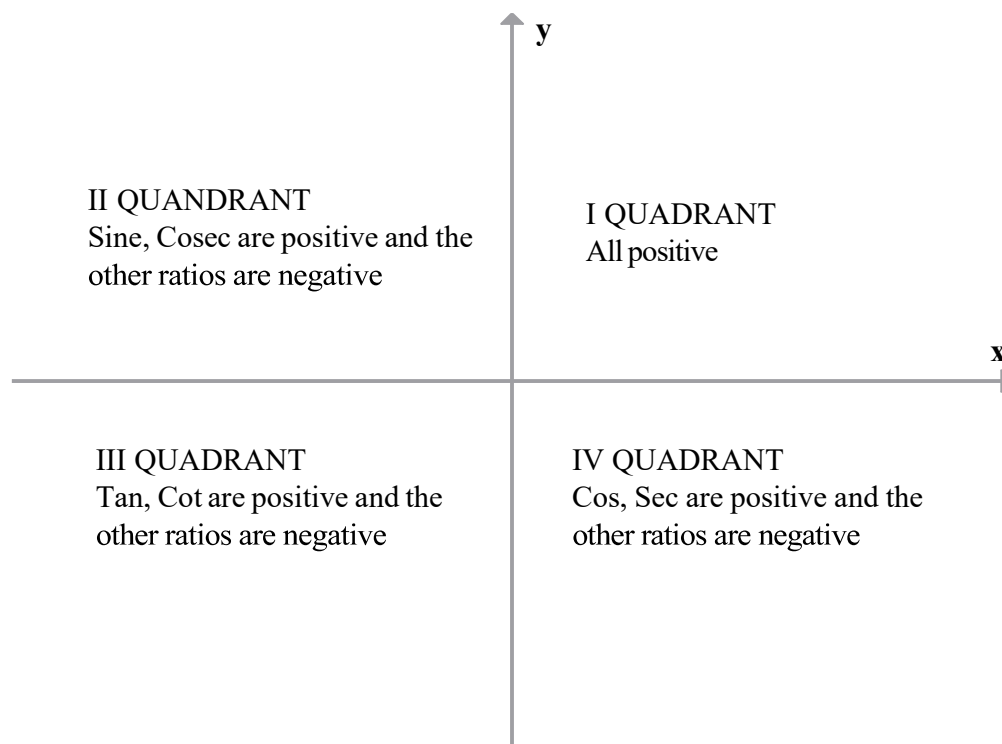
12. Cosine is even and sine is an odd function
    cos(−x) = cos x
    sin(−x) = −sin x

13. Signs of trigonometric functions in various quadrants
    In quadrant I, all the trigonometric functions are positive.
    In quadrant II, only sine is positive. In quadrant III, only tan is positive. In quadrant IV, only cosine function is positive. This is depicted as follows

y

x

II QUANDRANT
Sine, Cosec are positive and the
other ratios are negative

I QUADRANT
All positive

III QUADRANT
Tan, Cot are positive and the
other ratios are negative

IV QUADRANT
Cos, Sec are positive and the
other ratios are negative

14. In quadrants where Y-axis is positive (i.e. I and II), sine is positive, and in quadrants where X-axis is positive (i.e. I and IV), cosine is positive.

15. A simple rule to remember the signs of trigonometrical ratios in all the four quadrants is the four letter phrase—**A**ll **S**chool **T**o **C**ollege.

16. A function f is said to be a periodic function if there exists a real number T>0 such that f(x + T) = f(x) for all x. This T is the period of function.

17. Trigonometric ratios of complementary angles
    i. $\sin(90° - \theta) = \cos\theta$
    ii. $\cos(90° - \theta) = \sin\theta$
    iii. $\text{ten}(90° - \theta) = \cot\theta$
    iv. $\csc(90° - \theta) = \sec\theta$
    v. $\sec(90° - \theta) = \csc\theta$
    vi. $\cot(90° - \theta) = \tan\theta$

18. Trigonometric ratios of $(90° + \theta)$ in terms of θ
    i. $\sin(90° + \theta) = \cos\theta$
    ii. $\cos(90° + \theta) = -\sin\theta$
    iii. $\text{ten}(90° + \theta) = -\cot\theta$
    iv. $\csc(90° + \theta) = \sin\theta$

    v. $\sec(90^o + \theta) = \csc\theta$

    vi. $\cot(90^o + \theta) = -\tan\theta$

19. Trigonometric ratios of $(180^o - \theta)$ in terms of θ

    i. $\sin(180^o - \theta) = \sin\theta$

    ii. $\cos(180^o - \theta) = -\cos\theta$

    iii. $\text{ten}(180^o - \theta) = -\tan\theta$

    iv. $\csc(180^o - \theta) = \csc\theta$

    v. $\sec(180^o - \theta) = -\sec\theta$

    vi. $\cot(180^o - \theta) = -\cot\theta$

20. Trigonometric ratios of $(180^o + \theta)$ in terms of θ

    i. $\sin(180^o + \theta) = -\sin\theta$

    ii. $\cos(180^o + \theta) = -\cos\theta$

    iii. $\text{ten}(180^o + \theta) = \tan\theta$

    iv. $\csc(180^o + \theta) = -\csc\theta$

    v. $\sec(180^o + \theta) = -\sec\theta$

    vi. $\cot(180^o + \theta) = \cot\theta$

21. Trigonometric ratios of $(360^o - \theta)$ in terms of θ

    i. $\sin(360^o - \theta) = -\sin\theta$

    ii. $\cos(360^o - \theta) = \cos\theta$

    iii. $\text{ten}(360^o - \theta) = -\tan\theta$

    iv. $\csc(360^o - \theta) = -\csc\theta$

    v. $\sec(360^o - \theta) = \sec\theta$

    vi. $\cot(360^o - \theta) = -\cot\theta$

22. Trigonometric ratios of $(360^o + \theta)$ in terms of θ

    i. $\sin(360^o + \theta) = \sin\theta$

    ii. $\cos(360^o + \theta) = \cos\theta$

    iii. $\text{ten}(360^o + \theta) = \tan\theta$

    iv. $\csc(360^o + \theta) = -\csc\theta$

    v. $\sec(360^o + \theta) = \sec\theta$

    vi. $\cot(360^o + \theta) = \cot\theta$

23. In the case of $\sin(2\pi + x) = \sin x$, so the period of sine is 2π. Period of its reciprocal is also

2π.

24. In the case of cos $(2\pi + x)$ = cos x, so the period of cos is 2π. Period of its reciprocal is also 2π.

25. In the case of tan $(\pi + x)$ = tan x. Period of tangent and cotangent function is π.

26. The graph of cos x can be obtained by shifting the sine function by the factor $\dfrac{\pi}{2}$.

27. The tan function differs from the previous two functions in two ways

 (i) Function tan is not defined at the odd multiples of $\dfrac{\pi}{2}$.

 (ii) The tan function is not bounded.

28. | **Function** | **Period** |
|---|---|
| y = sin x | 2π |
| y = sin (ax) | $\dfrac{2\pi}{a}$ |
| y = cos x | 2π |
| y = cos (ax) | $\dfrac{2\pi}{a}$ |
| y = cos 3x | $\dfrac{2\pi}{3}$ |
| y = sin 5x | $\dfrac{2\pi}{5}$ |

29. For a function of the form

   y= kf(ax + b)

   the range will be k times the range of function x, where k is any real number.

   If f(x) = sine or cosine function, the range will be equal to R−[−k, k].
   If the function is of the form sec x or cosec x, the period is equal to the period of function f by a.
   The position of the graph is b units to the right/left of y = f(x) depending on whether b>0 or b<0.

30. The solutions of a trigonometric equation, for which $0 \leq x \leq 2\pi$, are called principal solutions.

29. The expression involving integer 'n' which gives all solutions of a trigonometric equation
    is called thegeneral solution.

30. The numerical smallest value of the angle (in degree or radian) satisfying a given trigonometric equation is called the Principal Value. If there are two values, one positive and the other negative, which are numerically equal, then the positive value is taken as the Principal Value.

Top Formulae

1. 1 radian = $\dfrac{180^0}{\pi}$ = 57°16' approximately.

2. 1° = $\dfrac{180^0}{\pi}$ radians = 0.01746 radians approximately.

3.

s= r θ

Length of arc = radius × angle in radian.

This relation can only be used when θ is in radians.

3. Radian measure = $\dfrac{\pi}{180}$ × degree measure.

4. degree measure = $\dfrac{180}{\pi}$ × Radian measure.

5. Values of trigonometric ratios:

	0°	$\dfrac{\pi}{6}$	$\dfrac{\pi}{4}$	$\dfrac{\pi}{3}$	$\dfrac{\pi}{2}$	π	$\dfrac{3\pi}{2}$	2π
sin	0	$\dfrac{1}{2}$	$\dfrac{1}{\sqrt{2}}$	$\dfrac{\sqrt{3}}{2}$	1	0	−1	0
cos	1	$\dfrac{\sqrt{3}}{2}$	$\dfrac{1}{\sqrt{2}}$	$\dfrac{1}{2}$	0	−1	0	1
tan	0	$\dfrac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined	0	Not defined	0
cosec	Not Defined	2	$\sqrt{2}$	$\dfrac{2}{\sqrt{3}}$	1	Not defined	−1	Not defined
sec	1	$\dfrac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined	−1	Not defined	1
cot	Not defined	$\sqrt{3}$	1	$\dfrac{1}{\sqrt{3}}$	0	Not defined	0	Not defined

6. Domain and range of various trigonometric functions:

Function	Domain	Range
y = sin x	$\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$	[−1, 1]
y = cos x	$[0, \pi]$	[−1, 1]
y = cosec x	$\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] - \{0\}$	R − (−1,1)
y = sec x	$[0, \pi] - \left\{\dfrac{\pi}{2}\right\}$	R − (−1, 1)
y = tan x	$\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$	R
y = cot x	$(0, \pi)$	R

7. Sign convention

	I	II	III	IV
sin x	+	+	—	—
cos x	+	—	—	+
tan x	+	—	+	—
cosec x	+	+	—	—
sec x	+	—	—	+
cot x	+	—	+	—

8. Behavior of Trigonometric Functions in various Quadrants

	I quadrant	II quadrant	III quadrant	IV quadrant
sin	Increases from 0 to 1	Decreases from 1 to 0	Decreases from 0 to −1	Increases from −1 to 0
cos	Decreases from 1 to 0	Decreases from 0 to −1	Increases from −1 to 0	Increases from 0 to 1
tan	Increases from 0 to ∞	Increases from −∞ to 0	Increases from 0 to ∞	Increases from −∞ to 0
cot	Decreases from ∞ to 0	Decreases from 0 to −∞	Decreases from ∞ to 0	Decreases from 0 to −∞
sec	Increases from 1 to ∞	Increases from −∞ to −1	Decreases from −1 to −∞	Decreases from ∞ to 1
cosec	Decreases from ∞ to 1	Increases from 1 to ∞	Increases from −∞ to −1	Decreases from −1 to −∞

9. Basic Formulae

(i) $\cos(x+y) = \cos x \cos y - \sin x \sin y$

(ii) $\cos(x-y) = \cos x \cos y + \sin x \sin y$

(iii) $\sin(x+y) = \sin x \cos y + \cos x \sin y$

(iv) $\sin(x-y) = \sin x \cos y - \cos x \sin y$

If none of the angles x, y and (x + y) is an odd multiple of $\dfrac{\pi}{2}$, then

(v) $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$

(vi) $\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$

If none of the angles x, y and (x + y) is a multiple of π, then

(vii) $\cot(x+y) = \dfrac{\cot x \cot y - 1}{\cot y + \cot x}$

(viii) $\cot(x-y) = \dfrac{\cot x \cot y + 1}{\cot y - \cot x}$

10. Allied Angle Relations

$\cos(2\pi - x) = \cos x$ $\sin(2\pi - x) = -\sin x$

$\cos(2n\pi + x) = \cos x$ $\sin(2n\pi + x) = \sin x$

11. Some Important Results

i. $\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y = \cos^2 y - \cos^2 x$

ii. $\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y = \cos^2 y - \sin^2 x$

iii. $\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y = \cos^2 y - \cos^2 x$

iv. $\sin(x+y+z)$

$= \sin x \cos y \cos z + \cos x \sin y \cos c + \cos x \cos y \sin z - \sin x \sin y \sin z$

v. $\cos(x+y+z)$

$= \cos x \cos y \cos z - \sin x \sin y \cos z - \sin x \cos y \sin z - \cos x \sin y \sin z$

vi. $\tan(x+y+z) = \dfrac{\tan x + \tan y + \tan z - \tan x \tan y \tan z}{1 - \tan x \tan y - \tan y \tan z - \tan z \tan x}$

12. Sum and Difference Formulae

(i) $\cos x + \cos y = 2\cos\dfrac{x+y}{2}\cos\dfrac{x-y}{2}$

(ii) $\cos x - \cos y = -2\sin\dfrac{x+y}{2}\sin\dfrac{x-y}{2}$

(iii) $\sin x + \sin y = 2\sin\dfrac{x+y}{2}\cos\dfrac{x-y}{2}$

(iv) $\sin x - \sin y = 2\cos\dfrac{x+y}{2}\sin\dfrac{x-y}{2}$

(v) $2\cos x \cos y = \cos(x+y) + \cos(x-y)$

(vi) $-2\sin x \sin y = \cos(x+y) - \cos(x-y)$

(vii) $2\sin x \cos y = \sin(x+y) + \sin(x-y)$

(viii) $2\cos x \sin y = \sin(x+y) - \sin(x-y)$

13. Multiple Angle Formulae

(i) $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$

(ii) $\sin 2x = 2\sin x \cos x = \dfrac{2\tan x}{1+\tan^2 x}$

(iii) $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

(iv) $\sin 3x = 3\sin x - 4\sin^3 x$

(v) $1 + \cos 2x = 2\cos^2 x$

(vi) $1 - \cos 2x = 2\sin^2 x$

(vi) $\cos 3x = 4\cos^3 x - 3\cos x$

(vii) $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

(viii) $\cos x \cdot \cos 2x \cdot \cos 2^2 x \cdot \cos 2^3 x \cdot \ldots \cdot \cos 2^{n-1}x = \dfrac{\sin 2^n x}{2^n \sin x}$

(ix) Let $x = \dfrac{\pi}{2^n + }$ then we have $2^n \cos x \cdot \cos 2x \cdot \cos 2^2 x \cdot \cos 2^3 x \cdot \ldots \cdot \cos 2^{n-1} x = 1$

(x) $\sin x \cdot \sin(60° - x) \cdot \sin(60° - x) = \dfrac{\sin 3x}{4}$

(xi) $\cos x \cdot \cos(60° - x) \cdot \cos(60° - x) = \dfrac{\cos 3x}{4}$

(xii) $(1 + \sec 2x)(1 + \sec 4x)(1 + \sec 8x)\ldots(1 + \sec 2^n x) = \tan 2^n x \cot x$

14. Trigonometric ratios of angle in terms of half angle

    i. $\sin x = 2\sin \dfrac{x}{2} \cos \dfrac{x}{2}$

    ii. $\cos x = \cos^2 \dfrac{x}{2} - \sin^2 \dfrac{x}{2}$

    iii. $\cos x = 2\cos^2 \dfrac{x}{2} - 1$

    iv. $\cos x = 1 - 2\sin^2 \dfrac{x}{2}$

    v. $1 + \cos x = 2\cos^2 \dfrac{x}{2}$

    vi. $1 - \cos x = 2\sin^2 \dfrac{x}{2}$

    vii. $\tan x = \dfrac{2\tan \dfrac{x}{2}}{1 - \tan^2 \dfrac{x}{2}}$

    viii. $\sin x = \dfrac{2\tan \dfrac{x}{2}}{1 + \tan^2 \dfrac{x}{2}}$

    ix. $\cos x = \dfrac{1 - \tan^2 \dfrac{x}{2}}{1 + \tan^2 \dfrac{x}{2}}$

    x. $\cos \dfrac{x}{2} = \pm\sqrt{\dfrac{1 + \cos x}{}}$

    xi. $\sin \dfrac{x}{2} = +\sqrt{\dfrac{1 - \cos x}{}}$

    xii. $\tan \dfrac{x}{2} = \pm\sqrt{\dfrac{1 - \cos x}{1 + \cos x}}$

15. Trigonometrical ratios of angle in terms of $\frac{x}{3}$ angle:

   i. $\sin x = 3\sin\frac{x}{3} - 4\sin^3\frac{x}{3}$

   ii. $\cos x = 4\cos^3\frac{x}{3} - 3\cos\frac{x}{3}$

   iii. $\tan x = \dfrac{3\tan\frac{x}{3} - \tan^3\frac{x}{3}}{1 - 3\tan^2\frac{x}{3}}$

16. Trigonometrical ratios of important angles:

   i. $\sin 18° = \dfrac{\sqrt{5}-1}{4}$

   ii. $\cos 36° = \dfrac{\sqrt{5}+1}{4}$

   iii. $\cos 18° = \dfrac{\sqrt{10+2\sqrt{5}}}{4}$

   iv. $\sin 36° = \dfrac{\sqrt{10-2\sqrt{5}}}{4}$

17. Trigonometric equations:

No.	Equations	General solution	Principal value
1	$\sin\theta = 0$	$\theta = n\pi, n\in Z$	$\theta = 0$
2	$\cos\theta = 0$	$\theta = (2n+1)\frac{\pi}{2}, n\in Z$	$\theta = \frac{\pi}{2}$
3	$\tan\theta = 0$	$\theta = n\pi$	$\theta = 0$
4	$\sin\theta = \sin\alpha$	$\theta = n\pi + (-1)^n \alpha$ $n\in Z$	$\theta = \alpha$
5	$\cos\theta = \cos\alpha$	$\theta = 2n\pi \pm \alpha \ n\in Z$	$\theta = 2\alpha, \alpha > 0$
6	$\tan\theta = \tan\alpha$	$\theta = n\pi + \alpha \ n\in Z$	$\theta = \alpha$
7	$\sin\ \theta = \sin^2\alpha$	$\theta = n\pi \pm \alpha, n\in Z$	
8	$\cos\ \theta = \cos^2\alpha$	$\theta = n\pi \pm \alpha, n\in Z$	
9	$\tan\ \theta = \tan^2\alpha$	$\theta = n\pi \pm \alpha, n\in Z$	

18. The equation $a\cos\theta + b\sin\theta = c$ is solvable for $|c| \le \sqrt{a^2\ b^2}$.

19. (i) $\sin\theta = k = \sin(n\pi + (-1)^n \alpha), n \in Z$

$\theta = n\pi + (-1)^n \alpha, n \in Z$

$\operatorname{cosec} \theta = \operatorname{cosec} \alpha \implies \sin \theta = \sin \alpha$

$\theta = n\pi + (-1)^n \alpha, n \in Z$

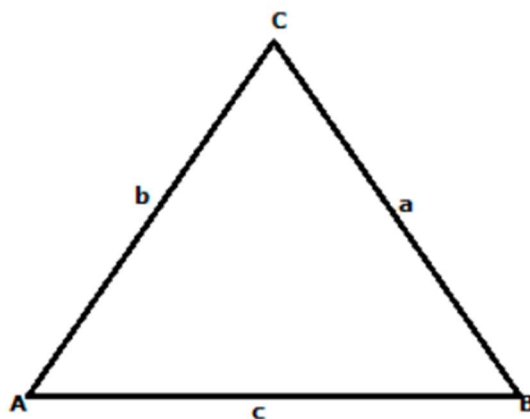(ii) $\cos \theta = k = \cos(2n\pi \pm \alpha), n \in$

$Z\theta = 2n\pi \pm \alpha, n \in Z$

20. Sine Rule: The sine rule states that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$
Or
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

21. Law of Cosine

In any $\triangle ABC$, $a^2 = b^2 + c^2 - 2bc\cos A$, $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$b^2 = c^2 + a^2 - 2ac\cos B$, $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

$a^2 = b^2 + c^2 - 2bc\cos A$, $\cos C = \dfrac{b^2 + c^2 - a^2}{2bc}$

22. Projection Formulae:

   (i) $a = b\cos C + c\cos B$

   (ii) $b = c\cos A + a\cos C$

   (iii) $c = a\cos B \ b\cos A$

23. Napier's Analogy (Law of Tangents)

(i) $\tan\left(\dfrac{B-C}{2}\right) = \left(\dfrac{b-c}{b+c}\right)\cot\dfrac{A}{2}$

(ii) $\tan\left(\dfrac{A-B}{2}\right) = \left(\dfrac{a-b}{a+b}\right)\cot\dfrac{C}{2}$

(iii) $\tan\left(\dfrac{C-A}{2}\right) = \left(\dfrac{c-a}{c+a}\right)\cot\dfrac{B}{2}$
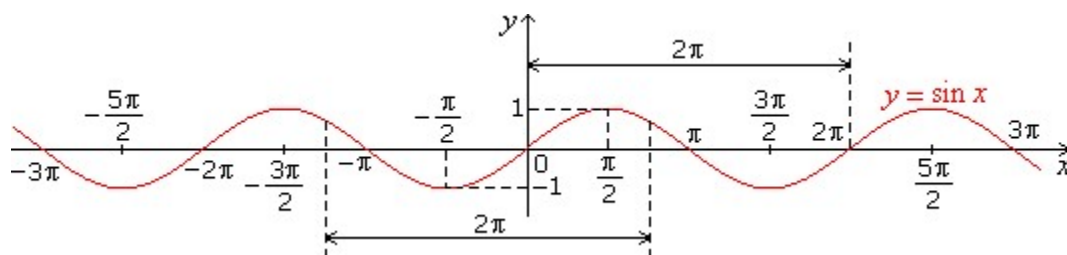
24. Area of ΔABC is given by

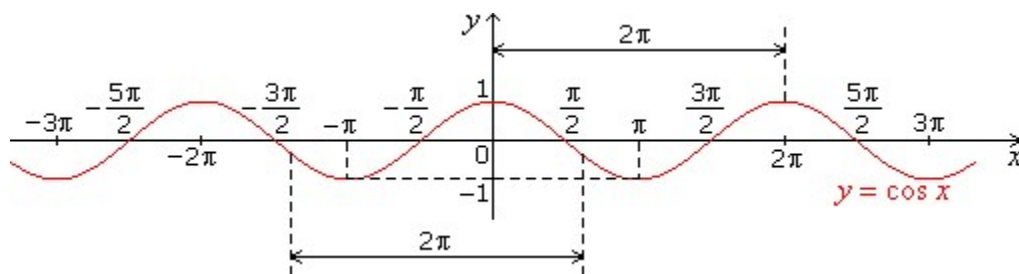$\Delta = \dfrac{1}{2} bc \sin A = \dfrac{1}{2} ca \sin B = \dfrac{1}{2} ab \sin C$

Top diagrams

1. Graphs helps in visualization of properties of trigonometric functions. The graph of y = sinθ can be drawn by plotting a number of points (θ, sin θ) as θ takes a series of different values. Because the sine function is continuous, these points can be joined with a smooth curve. Following similar procedures, graphs of other functions can be obtained.
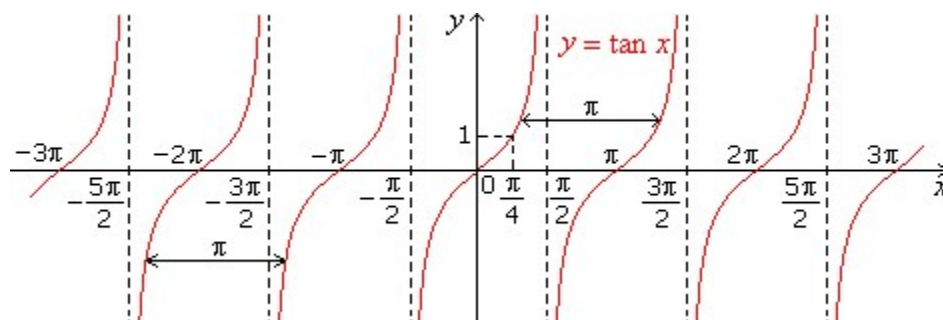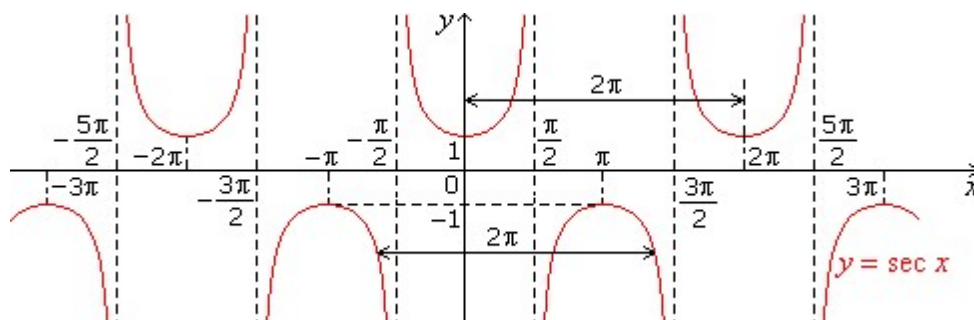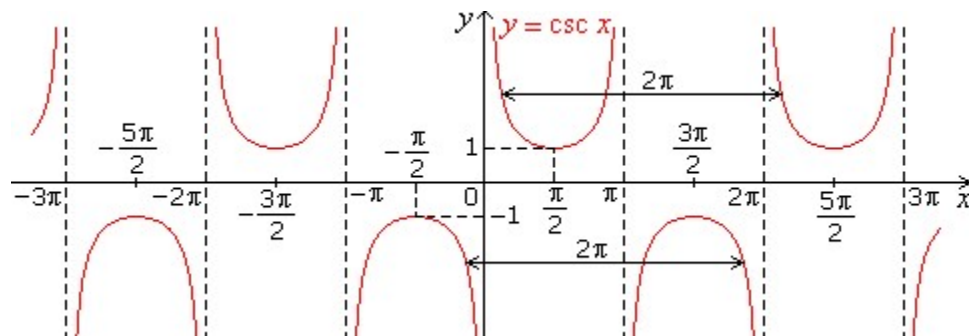
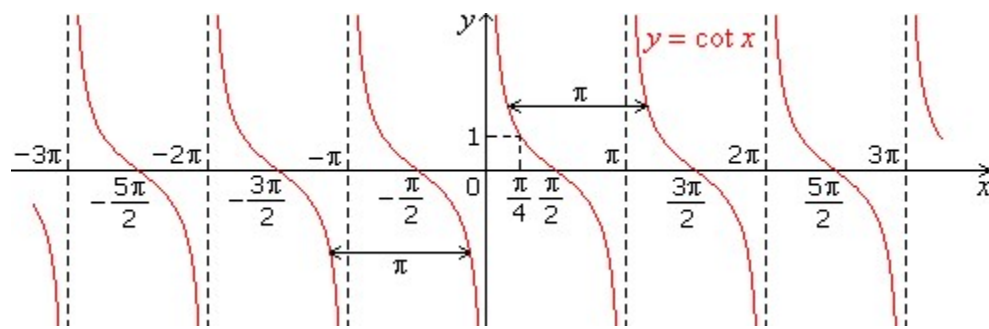i. Graph of sin x

ii. Graph of cos x

iii.   Graph of tan x

iv.   Graph of sec x

v. Graph of cosec x

vi.   Graph of cot x

MIND MAP : LEARNING MADE SIMPLE CHAPTER - 3

Trigonometric Functions

Trigonometrical function Sum and Difference of Two Angles

- $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
- $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
- $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$
- $\cot(A \pm B) = \dfrac{\cot A \cot B \mp 1}{\cot A \pm \cot B}$
- $\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A = \sin(A+B)\sin(A-B)$
- $\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A = \cos(A+B)\cos(A-B)$

Transformation of Product into Sum or Difference of Sines or Cosines

- $2\sin A \sin B = \sin(A+B) + \sin(A-B)$
- $2\cos A \sin B = \sin(A+B) - \sin(A-B)$
- $2\cos A \cos B = \cos(A+B) + \cos(A-B)$
- $2\sin A \sin B = \cos(A-B) - \cos(A+B)$

Degree & Radian Measures

If in a circle of radius r, an arc of length l subtends an angle of θ radians, then $l = r\theta$.
- Radian Measure $= \dfrac{\pi}{180} \times$ Degree Measure
- Degree Measure $= \dfrac{180}{\pi} \times$ Radian Measure

Trigonometric Identities

- $\sin^2\theta + \cos^2\theta = 1, -1 \le \sin\theta \le 1, -1 \le \cos\theta \le 1 \ \forall\theta\in R$
- $\sec^2\theta - \tan^2\theta = 1, |\sec\theta| \ge 1, \forall\theta\in R$
- $\cosec^2\theta - \cot^2\theta = 1, |\cosec\theta| \ge 1, \theta\in R$

Trigonometrical Ratio of allied Angles

- $\sin(-\theta) = -\sin\theta$
- $\cos(-\theta) = \cos\theta$
- $\tan(-\theta) = -\tan\theta$
- $\cot(-\theta) = -\cot\theta$
- $\sec(-\theta) = \sec\theta$
- $\cosec(-\theta) = -\cosec\theta$

- $\sin\left(\dfrac{\pi}{2} \mp \theta\right) = \cos\theta$
- $\cos\left(\dfrac{\pi}{2} \mp \theta\right) = \pm\sin\theta$
- $\tan\left(\dfrac{\pi}{2} \mp \theta\right) = \pm\cot\theta$
- $\cot\left(\dfrac{\pi}{2} \mp \theta\right) = \pm\tan\theta$
- $\sec\left(\dfrac{\pi}{2} \mp \theta\right) = \pm\cosec\theta$
- $\cosec\left(\dfrac{\pi}{2} \mp \theta\right) = \sec\theta$

- $\cos((\pi \mp \theta) = -\cos\theta$
- $\sin(\pi \mp \theta) = \pm\sin\theta$
- $\tan(\pi \mp \theta) = \mp\tan\theta$
- $\cot(\pi \mp \theta) = \mp\cot\theta$
- $\sec(\pi \mp \theta) = -\sec\theta$
- $\cosec(\pi \mp \theta) = \pm\cosec\theta$

- $\sin\left(\dfrac{3\pi}{2} \mp \theta\right) = -\cos\theta$
- $\cos\left(\dfrac{3\pi}{2} \mp \theta\right) = \mp\sin\theta$
- $\tan\left(\dfrac{3\pi}{2} \mp \theta\right) = \pm\cot\theta$
- $\cot\left(\dfrac{3\pi}{2} \mp \theta\right) = \pm\tan\theta$
- $\sec\left(\dfrac{3\pi}{2} \mp \theta\right) = \mp\cosec\theta$
- $\cosec\left(\dfrac{3\pi}{2} \mp \theta\right) = \sec\theta$

- $\sin(2\pi \mp \theta) = \mp\sin\theta$
- $\cos(2\pi \mp \theta) = \cos\theta$
- $\tan(2\pi \mp \theta) = \mp\tan\theta$
- $\cot(2\pi \mp \theta) = \mp\cot\theta$
- $\sec(2\pi \mp \theta) = \sec\theta$
- $\cosec(2\pi \mp \theta) = \mp\cosec\theta$

Multiple and Half Angles

- $\sin 2A = 2\sin A \cos A$
- $\cos 2A = \cos^2 A - \sin^2 A$
- $\quad = 2\cos^2 A - 1$
- $\quad = 1 - 2\sin^2 A$
- $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$
- $\sin 2A = \dfrac{2\tan A}{1 + \tan^2 A}$
- $\cos 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$
- $\sin 3A = 3\sin A - 4\sin^3 A$
- $\cos 3A = 4\cos^3 A - 3\cos A$
- $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$
- $\sin\theta = 2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}$
- $\cos\theta = 2\cos^2\dfrac{\theta}{2} - 1 = 1 - 2\sin^2\dfrac{\theta}{2}$
- $\tan\theta = 2\tan\dfrac{\theta}{2}\left(1 - \tan^2\dfrac{\theta}{2}\right)$

Trigonometrical Equations

The equation involving trigonometric functions of unknown angles are known as Trigonometric equations.
e.g $\cos\theta = 0, \cos^2\theta - 4\cos\theta = 1$
A solution of trigonometric equation is the value of the unknown angle that satisfies the equation.

e.g. $\sin\theta = \dfrac{1}{\sqrt{2}}$

$\Rightarrow \theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{9\pi}{4}, \dfrac{11\pi}{4}, \ldots$

Standard General Solutions of Trigonometrical ratios

General solution: The solution consisting of all possible solutions of a trigonometric equation is called its General Solution.

- $\sin\theta = 0 \Leftrightarrow \theta = n\pi, \cos\theta = 0 \Leftrightarrow \theta = (2n + \dfrac{\pi}{2})$
- $\tan\theta = 0 \Leftrightarrow \theta = n\pi, \sin\theta = \sin\alpha \Leftrightarrow \theta = n\pi + (-1)^n \alpha$, where $\alpha \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$
- $\cos\theta = \cos\alpha \Leftrightarrow \theta = 2n\pi \pm \alpha$, where $\alpha \in [0, \pi]$
- $\tan\theta = \tan\alpha \Leftrightarrow \theta = n\pi + \alpha$, where $\alpha \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$
- $\sin^2\theta = \sin^2\alpha \Leftrightarrow \theta = n\pi \pm \alpha, \cos^2\theta = \cos^2\alpha \Leftrightarrow \theta = n\pi \pm \alpha$
- $\tan^2\theta = \tan^2\alpha \Leftrightarrow \theta = n\pi \pm \alpha, \sin\theta = 1 \Leftrightarrow \theta = (4n+1)\dfrac{\pi}{2}$
- $\cos\theta = 1 \Leftrightarrow \theta = 2n\pi, \cos = -1 \Leftrightarrow \theta = (2n+1)\pi$
- $\sin\theta = \sin\alpha$ and $\cos\theta = \cos\alpha \Leftrightarrow \theta = 2n\pi + \alpha, \eta\in z$

Important Trigonometric Ratios values

- $\sin 15°$ or $\sin\dfrac{\pi}{12} = \dfrac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75°$ or $\cos\dfrac{5\pi}{12}$
- $\cos 15°$ or $\cos\dfrac{\pi}{12} = \dfrac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75°$ or $\sin\dfrac{5\pi}{12}$
- $\tan 15° = \dfrac{\sqrt{3}-1}{\sqrt{3}+1} = 2 - \sqrt{3} = \cot 75°$
- $\tan 75° = \dfrac{\sqrt{3}+1}{\sqrt{3}-1} = 2 + \sqrt{3} = \cot 15°$
- $\sin\dfrac{\pi}{10}$ or $\sin 18° = \dfrac{\sqrt{5}-1}{4}$
- $\cos\dfrac{\pi}{5}$ or $\cos 36° = \dfrac{\sqrt{5}+1}{4}$

Factorisation of the Sum or Difference of sine and cosine with two variables

- $\sin C + \sin D = 2\sin\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$
- $\sin C - \sin D = 2\cos\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)$
- $\cos C + \cos D = 2\cos\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$
- $\cos C - \cos D = -2\sin\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)$

Important Questions

Multiple Choice questions-

Question 1. The value of sin 15 + cos 15 is

(a) 1

(b) 1/2

(c) √3/2

(d) √3

Question 2. The value of tan A/2 – cot A/2 + 2cot A is

(a) 0

(b) 1

(c) -1

(d) None of these

Question 3. The value of 4 × sin x × sin(x + π/3) × sin(x + 2π/3) is

(a) sin x

(b) sin 2x

(c) sin 3x

(d) sin 4x

Question 4. If tan x = (cos 9 + sin 9)/(cos 9 – sin 9), then x =

(a) 45

(b) 54

(c) 36

(d) None of these

Question 5. In a triangle ABC, sin A – cos B = cos C, then angle B is

(a) π/2

(b) π/3

(c) π/4

(d) π/6

Question 6. The value of cos 420° is

(a) 0

(b) 1

(c) 1/2

(d) √3/2

Question 7. If in a triangle ABC, tan A + tan B + tan C = 6 then the value of cot A × cot B × cot C is

(a) 1/2

(b) 1/3

(c) 1/4

(d) 1/6

Question 8. If a × cos x + b × cos x = c, then the value of (a × sin x − b × cos x)² is

(a) a² + b² + c²

(b) a² − b² − c²

(c) a² − b² + c²

(d) a² + b² − c²

Question 9. When the length of the shadow of a pole is equal to the height of the pole, then the elevation of source of light is

(a) 30°

(b) 60°

(c) 75°

(d) 45°

Question 10. In any triangle ABC, if cos A/a = cos B/b = cos C/c and the side a = 2, then the area of the triangle is

(a) √3

(b) √3/4

(c) √3/2

(d) 1/√3

Very Short:

1. Find the radian measure corresponding to 5° 37" 30""

2. Find the degree measure corresponding to $\left(\frac{11}{16}\right)^c$.

3. Find the length of an arc of a circle of radius 5 cm subtending a central angle measuring 15°

4. Find the value of $\frac{19\pi}{3}$.

5. Find the value of $\sin(-1125°)$

6. Find the value of $\tan 15°$

7. If $\sin A = \dfrac{3}{5}$ and $\dfrac{\pi}{2} < A < \pi$, find $\cos A$

8. If $\tan A = \dfrac{a}{a+1}$ and $\tan B = \dfrac{1}{2a+1}$ then find the value of A + B.

9. Express $\sin 12\theta + \sin 4\theta$ as the product of sines and cosines.

10. Express $2 \cos 4x \sin 2x$ as an algebraic sum of sines or cosines.

Short Questions:

1. The minute hand of a watch is 1.5 cm long. How far does it tip move in 40 minute?

2. Show that $\tan 3x \cdot \tan 2x \cdot \tan x = \tan 3x - \tan 2x - \tan x$

3. Find the value of $\tan \dfrac{\pi}{8}$.

4. Prove that $\dfrac{\sin (x+y)}{\sin (x-y)} = \dfrac{\tan x + \tan y}{\tan x - \tan y}$

5. If in two circles, arcs of the same length subtend angles 600 and 750 at the center find the ratio of their radii.

Long Questions:

1. If $\sin \alpha + \sin \beta = a$ and $\cos \alpha + \cos \beta = b$ show that $\cos (\alpha + \beta) = \dfrac{b^2 - a^2}{b^2 + a^2}$

2. Prove that $\cos \alpha + \cos \beta + \cos \gamma + \cos (\alpha + \beta + \gamma)$

$$= 4 \cos \left(\dfrac{\alpha + \beta}{2}\right) \cdot \cos \left(\dfrac{\beta + \gamma}{2}\right) \cdot \cos \left(\dfrac{\gamma + \alpha}{2}\right)$$

3. Prove that $\sin 3x + \sin 2x - \sin 2x = 4 \sin x \cdot \cos \dfrac{x}{2} \cdot \cos \dfrac{3x}{2}$

4. Prove that $2\cos \dfrac{\pi}{13} \cdot \cos \dfrac{9\pi}{13} + \cos \dfrac{3\pi}{13} + \cos \dfrac{5\pi}{13} = 0$

5. Find the value of $\tan (\alpha + \beta)$ Given that.

$$\cot \alpha = \dfrac{1}{2}, \alpha \in \left(\pi, \dfrac{3\pi}{2}\right) \text{ and } \sec \beta = -\dfrac{5}{3}, \beta \in \left(\dfrac{\pi}{2}, \pi\right)$$

Assertion Reason Questions:

1. In the following questions, A statement of Assertion (A) is followed by a statement of Reason (R). Mark the correct choice as.

Assertion (A) : $\sin^{-1}(\sin(2\pi/3)) = 2\pi/3$

Reason (R) : $\sin^{-1}(\sin \theta) = \theta$, if $q \in [(-\pi)/2, \pi/2]$

(i) Both A and R are true and R is the correct explanation of A

(ii) Both A and R are true but R is NOT the correct explanation of A

(iii) A is true but R is false

(iv) A is false and R is True

2. In the following questions, A statement of Assertion (A) is followed by a statement of Reason (R). Mark the correct choice as.

Assertion (A) : Principal value of $\cos^{-1}(1)$ is π

Reason (R) : Value of cos 0° is 1

(i) Both A and R are true and R is the correct explanation of A

(ii) Both A and R are true but R is NOT the correct explanation of A

(iii) A is true but R is false

(iv) A is false and R is True

Answer Key:

MCQ

1. (c) √3/2

2. (a) 0

3. (c) sin 3x

4. (b) 54

5. (a) π/2

6. (c) 1/2

7. (d) 1/6

8. (d) a² + b² – c²

9. (d) 45°

10. (a) √3

Very Short Answer:

1. $\left(\dfrac{\pi}{32}\right)^c$

2. 39°22"30""

3. $\dfrac{5\pi}{12} \, cm$

4. $\sqrt{3}$

5. $\dfrac{-1}{\sqrt{2}}$

6. $2 - \sqrt{3}$

7. $\dfrac{-4}{5}$

8. $45°$

9. $2 \sin 8\theta \cos 4\theta$

10. $\sin 6x - \sin 2x$

Short Answer:

1. r = 1.5 cm

   Angle made in 60 mint = 360˚

   Angle made in 1 min = $\dfrac{360}{60}$ = 60˚

   Angle made in 40 mint = 6 × 40

   = 240˚

   $\Theta = \dfrac{l}{r}$

   $\cancel{240}^{\,4} \times \dfrac{\pi}{\cancel{180}\ 3} = \dfrac{l}{1.5}$

   $\dfrac{4 \times 3.14}{\cancel{3}} = \dfrac{l}{\dfrac{\cancel{1.5}\ \cancel{5}}{\cancel{10}\ 2}}$

   $2 \times 3.14 = l$

   $6.28 = l$

   $l = 6.28$ cm

2.

   Let 3x = 2x + x

   tan 3x = tan (2x + x)

   $\dfrac{\tan 3x}{1} = \dfrac{\tan 2x + \tan x}{1 - \tan 2x.\ \tan x}$

   tan 3x (1 - tan 2x. tan x) = tan 2x + tan x

   tan 3x - tan 3x. tan 2x. tan x = tan 2x + tan x

   tan 3x. tan 2x. tan x = tan 3x - tan 2x - tan x

3.

Let x = $\dfrac{\pi}{8}$

$\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x} t$

$\tan \left(2 \dfrac{\pi}{8 \ 4} \right) = \dfrac{2 \tan \pi/8}{1 - \tan^2 \pi/8}$

$1 = \dfrac{2 \tan \pi/8}{1 - \tan^2 \pi/8}$

put $\tan \pi/8 = $

$\dfrac{1}{1} = \dfrac{2t}{1 - t^2}$

$2t = 1 - t^2$

$t^2 + 2t - 1 = 0$

$t = \dfrac{-2 \pm 2\sqrt{2}}{2 \times 1}$

$= -1 \pm \sqrt{2}$

$= \pm \sqrt{2} - 1$

$= \sqrt{2} - 1 \text{ or } -\sqrt{2} - 1$

$\tan \pi/8 = \sqrt{2} - 1$

4.

L.H.S $= \dfrac{Sin(x+y)}{Sin(x-y)}$

$= \dfrac{Sinx.Cosy + Cosx.Siny}{Sinx.Cosy - Cosx.Siny}$

Dividing N and D by Cosx.Cosy

$= \dfrac{\tan x + \tan y}{\tan x - \tan y}$

5.

$$\theta = \frac{l}{r_1}$$

$$60 \times \frac{\pi}{18} = \frac{l}{r_1}$$

$$r_1 = \frac{3l}{\pi} \quad (1)$$

$$\theta = \frac{l}{r_2}$$

$$75 \times \frac{\pi}{18} = \frac{l}{r_2}$$

$$r_2 = \frac{12l}{5\pi} \quad (2)$$

$(1) \div (2)$

$$\frac{r_1}{r_2} = \frac{\dfrac{3l}{\pi}}{\dfrac{12l}{5\pi}}$$

$$= \frac{3l}{\pi} \times \frac{5\pi}{12l}$$

$$= 5:4$$

Long Answer:

1.

$$b^2 + a^2 = (\text{Cos } \alpha + \text{Cos } \beta)^2 + (\text{Sin } \alpha + \text{Sin } \beta)^2$$

$$= \text{Cos}^2 \alpha + \text{Cos}^2 \beta + 2 \text{ Cos } \alpha. \text{ Cos } \beta + \text{Sin}^2 \alpha + \text{Sin}^2 \beta + 2\text{Sin } \alpha. \text{ Sin } \beta$$

$$= 1 + 1 + 2 (\text{Cos } \alpha. \text{ Cos } \beta + \text{Sin } \alpha . \text{ Sin } \beta)$$

$$= 2 + 2 \text{ Cos } (\alpha - \beta) \quad (1)$$

$$b^2 - a^2 = (\text{Cos } \alpha + \text{Cos } \beta)^2 - (\text{Sin } \alpha + \text{Sin } \beta)^2$$

$$= (\text{Cos}^2 \alpha - \text{Sin}^2 \beta) + (\text{Cos}^2 \beta - \text{Sin}^2 \alpha) + 2 \text{ Cos } (\alpha + \beta)$$

$= \cos(\alpha + \beta)\ \cos(\alpha - \beta) + \cos(\beta + \alpha)\cos(\alpha - \beta) + 2\cos(\alpha + \beta)$

$= 2\cos(\alpha + \beta).\ \cos(\alpha - \beta) + 2\cos(\alpha + \beta)$

$= \cos(\alpha + \beta)\left[2\cos(\alpha - \beta) + 2\right]$

$= \cos(\alpha + \beta).\ (b^2 + a^2)\ [\text{from } (1)$

$\dfrac{b^2 - a^2}{b^2 + a^2} = \cos(\alpha + \beta)$

2. L. H. S.

$= \cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha + \beta + \gamma)$

$= \cos\alpha + \cos\beta) + \left[\cos\gamma + \cos(\alpha + \beta + \gamma)\right]$

$= 2\cos\left(\dfrac{\alpha + \beta}{2}\right).\ \cos\left(\dfrac{\alpha - \beta}{2}\right) + 2\cos\left(\dfrac{\alpha + \beta + \gamma + \gamma}{2}\right).\ \cos\left(\dfrac{\alpha + \beta + \gamma - \gamma}{2}\right)$

$= 2\cos\left(\dfrac{\alpha + \beta}{2}\right).\ \cos\left(\dfrac{\alpha - \beta}{2}\right) + 2\cos\left(\dfrac{\alpha + \beta}{2}\right).\ \cos\left(\dfrac{\alpha + \beta + 2\gamma}{2}\right)$

$= 2\cos\left(\dfrac{\alpha + \beta}{2}\right)\left[\cos\left(\dfrac{\alpha - \beta}{2}\right) + \cos\left(\dfrac{\alpha + \beta + 2\gamma}{2}\right)\right]$

$= 2\cos\left(\dfrac{\alpha + \beta}{2}\right)\left[2\cos\left(\dfrac{\frac{\alpha - \beta}{2} + \frac{\alpha + \beta + 2\gamma}{2}}{2}\right).\ \cos\left(\dfrac{\frac{\alpha + \beta + 2\gamma}{2} - \frac{\alpha - \beta}{2}}{2}\right)\right]$

$= 2\cos\left(\dfrac{\alpha + \beta}{2}\right)\left[2\cos\left(\dfrac{\alpha + \gamma}{2}\right).\ \cos\left(\dfrac{\beta + \gamma}{2}\right)\right]$

$= 4\cos\left(\dfrac{\alpha + \beta}{2}\right).\ \cos\left(\dfrac{\beta + \gamma}{2}\right).\ \cos\left(\dfrac{\gamma + \alpha}{2}\right)$

3.

$(\sin 3x - \sin x) + \sin 2x$

$= 2\cos\left(\dfrac{3x + x}{2}\right).\ \sin\left(\dfrac{3x - x}{2}\right) + \sin 2x$

$= 2\cos 2x.\sin x + \sin 2x$

$= 2\cos 2x.\sin x + 2\sin x \cos x$

$= 2\sin x\left[\cos 2x + \cos x\right]$

$= 2\sin x\left[2\cos x\dfrac{3x}{2}.\cos\dfrac{x}{2}\right]$

$$= 4\sin x \cos\frac{3x}{2}\cos\frac{x}{2}$$

4. L.H.S.

$$= 2\cos\frac{\pi}{13}.\cos\frac{9\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13}$$

$$= \cos\left(\frac{\pi}{13}+\frac{9\pi}{13}\right) + \cos\left(\frac{\pi}{13}-\frac{9\pi}{13}\right) + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13}$$

$$= \cos\frac{10\pi}{13} + \cos\frac{8\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13}$$

$$= \cos\left(\pi-\frac{3\pi}{13}\right) + \cos\left(\pi-\frac{5\pi}{13}\right) + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13}$$

$$= -\cancel{\cos\frac{3\pi}{13}} - \cancel{\cos\frac{5\pi}{13}} + \cancel{\cos\frac{3\pi}{13}} + \cancel{\cos\frac{5\pi}{13}}$$

$$= 0$$

5.

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha.\tan\beta} \quad (1)$$

$$\text{Cot }\alpha = \frac{1}{2},$$

$$\Rightarrow \tan\alpha = 2$$

$$1 + \tan^2\beta = \text{Sec}^2\beta$$

$$1 + \tan^2\beta = \left(\frac{-5}{3}\right)^2 \quad \left[\because \text{Sec }\beta = \frac{-5}{3}1\right)$$

$$\tan\beta = \pm\frac{4}{3}$$

$$\tan\beta = -\frac{4}{3} \quad \left[\because \beta \in \left(\frac{\pi}{2}/x\right)\right]$$

put $\tan\alpha$, and $\tan\beta$ in eq.

$$\tan(\alpha+\beta) = \frac{2-\frac{4}{3}}{1-2\left(-\frac{4}{3}\right)}$$

$$= \frac{2}{11}$$

Assertion Reason Answer:

1. (iv) A is false and R is True

2. (i) Both A and R are true and R is the correct explanation of A